

# On the separability criterion for continuous variable systems

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## Abstract

We present an elementary and explicit proof of the separability criterion for continuous variable two-party Gaussian systems. Our proof is based on an elementary formulation of uncertainty relations and an explicit determination of squeezing parameters for which the P-representation condition saturates the  $Sp(2, R) \otimes Sp(2, R)$  invariant separability condition. We thus give the explicit formulas of squeezing parameters, which establish the equivalence of the separability condition with the P-representation condition, in terms of the parameters of the standard form of the correlation matrix. Our proof is compared to the past proofs, and it is pointed out that the original proof of the P-representation by Duan, Giedke, Cirac and Zoller(DGCZ) is incomplete. A way to complete their proof is then shown. It is noted that both of the corrected proof of DGCZ and the proof of R. Simon are closely related to our explicit construction despite their quite different appearances.

## 1 Introduction

The entanglement [1] is an intriguing property of quantum mechanics, but a quantitative criterion of entanglement appears to be missing except for simple systems such as a two-spin system [2, 3]. In view of this situation, it is remarkable that a quantitative sufficient condition for continuous variable two-party systems exists and that the criterion is necessary and sufficient for Gaussian states [4, 5]. The proofs given by Duan, Giedke, Cirac and Zoller (DGCZ) [4] and Simon [5] which consist of a series of logical steps are ingenious. However, their proofs are based on some specific notions and ideas in quantum optics, and thus their proofs are not readily accessible to those physicists who are interested only in the general aspects of entanglement in quantum mechanics. Moreover, these two proofs are seemingly quite different and their mutual connections are not clear. This problem and related

issues have been discussed in the past by several authors [6, 7, 8, 9, 10, 11, 12, 13]. The present status of the quantum separability problem is nicely reviewed in [14].

We here present an elementary and explicit proof by starting with the elementary analysis of Heisenberg uncertainty relations in the manner of Kennard [15, 16] and an explicit determination of squeezing parameters which establish that the P-representation condition saturates the  $Sp(2, R) \otimes Sp(2, R)$  invariant separability condition. We thus give the explicit formulas of squeezing parameters, which establish the equivalence of the separability condition and the P-representation condition, in terms of the parameters of the standard form of the correlation matrix (or second moments). It is also pointed out that the original proof of the P-representation by DGCZ is incomplete, and a way to complete their proof is shown. It is then shown that both of the corrected proof of DGCZ and the seemingly quite different proof of Simon are closely related to our explicit construction.

Our treatment is based on a clear recognition that the separability condition associated with uncertainty relations is invariant under general  $Sp(2, R) \otimes Sp(2, R)$  transformations, whereas the condition for the P-representation of Gaussian states is not invariant under general  $Sp(2, R) \otimes Sp(2, R)$  transformations. A combination of these two apparently contradicting relations is the basis of our construction of the explicit solution.

## 2 Entanglement and Kennard's relation

### 2.1 Kennard's relation

We consider a two-party system (or a two-particle system in one-dimensional space) described by canonical variables  $(q_1, p_1)$  and  $(q_2, p_2)$ . We define

$$\begin{aligned}\hat{X}(d, f) &= d_1 \hat{q}_1 + d_2 \hat{p}_1 + f_1 \hat{q}_2 + f_2 \hat{p}_2, \\ \hat{X}(g, h) &= g_1 \hat{q}_1 + g_2 \hat{p}_1 + h_1 \hat{q}_2 + h_2 \hat{p}_2\end{aligned}\tag{2.1}$$

where all the coefficients

$$d^T = (d_1, d_2), \quad f^T = (f_1, f_2), \quad g^T = (g_1, g_2), \quad h^T = (h_1, h_2)\tag{2.2}$$

are real numbers. The Kennard's relation for a mixed state  $\hat{\rho} = \sum_k P_k |\psi_k\rangle \langle \psi_k|$  with  $P_k \geq 0$  and  $\sum_k P_k = 1$  is written as [5] (for any choice of  $d \sim h$ )

$$\langle (\Delta \hat{X}(d, f))^2 \rangle + \langle (\Delta \hat{X}(g, h))^2 \rangle \geq |d^T J g + f^T J h|\tag{2.3}$$

where we defined

$$\langle (\Delta \hat{X}(d, f))^2 \rangle = \text{Tr}\{(\Delta \hat{X}(d, f))^2 \hat{\rho}\}\tag{2.4}$$

with  $\Delta\hat{X}(d, f) = \hat{X}(d, f) - \langle\hat{X}(d, f)\rangle$  and  $\langle\hat{X}(d, f)\rangle = \text{Tr}\{\hat{X}(d, f)\hat{\rho}\}$ , for example, and the  $2 \times 2$  matrix

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.5)$$

The relation (2.3) is derived from  $\text{Tr}\{\hat{\eta}\hat{\eta}^\dagger\hat{\rho}\} \geq 0$  and  $\text{Tr}\{\hat{\eta}^\dagger\hat{\eta}\hat{\rho}\} \geq 0$  for  $\hat{\eta} = \Delta\hat{X}(d, f) + i\Delta\hat{X}(g, h)$ , and the right-hand side of (2.3) stands for  $||[\Delta\hat{X}(d, f), \Delta\hat{X}(g, h)]||$ .

We examine (2.3) more precisely by starting with

$$\begin{aligned} \langle(\Delta\hat{X}(d, f))^2\rangle &= \langle(\hat{X}(d, f) - \langle\hat{X}(d, f)\rangle)^2\rangle \\ &= \sum_k P_k \langle(\hat{X}(d, f) - \langle\hat{X}(d, f)\rangle)^2\rangle_k \\ &= \sum_k P_k \langle(\hat{X}(d, f) - \langle\hat{X}(d, f)\rangle_k + \langle\hat{X}(d, f)\rangle_k - \langle\hat{X}(d, f)\rangle)^2\rangle_k \\ &= \sum_k P_k [\langle(\hat{X}(d, f) - \langle\hat{X}(d, f)\rangle_k)^2\rangle_k + (\langle\hat{X}(d, f)\rangle_k - \langle\hat{X}(d, f)\rangle)^2] \\ &\geq \sum_k P_k \langle(\hat{X}(d, f) - \langle\hat{X}(d, f)\rangle_k)^2\rangle_k \end{aligned} \quad (2.6)$$

which holds for a general mixed state for any real numbers  $d$  and  $f$ . The equality sign holds only for

$$\langle\hat{X}(d, f)\rangle_k - \langle\hat{X}(d, f)\rangle = 0 \quad (2.7)$$

for all  $k$  where  $\langle\hat{X}(d, f)\rangle = \sum_k P_k \langle\hat{X}(d, f)\rangle_k$  with

$$\langle\hat{X}(d, f)\rangle_k = \int dq_1 dq_2 \psi_k^*(q_1, q_2) \hat{X}(d, f) \psi_k(q_1, q_2). \quad (2.8)$$

The condition (2.7) is trivial for a pure state, but it imposes a stringent condition on a mixed state. The Kennard's relation for a general pure state is given by (2.3) if one sets  $P_k = 1$  for specific  $k$  and others zero

$$\begin{aligned} &\langle(\hat{X}(d, f) - \langle\hat{X}(d, f)\rangle_k)^2\rangle_k + \langle(\hat{X}(g, h) - \langle\hat{X}(g, h)\rangle_k)^2\rangle_k \\ &\geq |d^T J g + f^T J h| \end{aligned} \quad (2.9)$$

for any  $d \sim h$ , which is also written as

$$\begin{aligned} &t^2 \langle(\hat{X}(d, f) - \langle\hat{X}(d, f)\rangle_k)^2\rangle_k + \langle(\hat{X}(g, h) - \langle\hat{X}(g, h)\rangle_k)^2\rangle_k \\ &- t |d^T J g + f^T J h| \geq 0 \end{aligned} \quad (2.10)$$

by replacing  $d \rightarrow td$ ,  $f \rightarrow tf$  for any real  $t$  and thus the discriminant gives the conventional form of Kennard's relation. The Kennard's relations for pure states imply

$$\begin{aligned} & \sum_k P_k [\langle (\hat{X}(d, f) - \langle \hat{X}(d, f) \rangle_k)^2 \rangle_k + \langle (\hat{X}(g, h) - \langle \hat{X}(g, h) \rangle_k)^2 \rangle_k] \\ & \geq |d^T Jg + f^T Jh| \end{aligned} \quad (2.11)$$

for any  $d \sim h$ , which is more precise than (2.3) because of the removal of extra terms as in (2.6).

## 2.2 Separability condition

For a separable pure state  $\psi_k(q_1, q_2) = \phi_k(q_1)\varphi_k(q_2)$ , we have

$$\begin{aligned} \langle (\hat{X}(d, f) - \langle \hat{X}(d, f) \rangle_k)^2 \rangle_k &= \langle (d_1 \hat{q}_1 + d_2 \hat{p}_1 - \langle d_1 \hat{q}_1 + d_2 \hat{p}_1 \rangle_k)^2 \rangle_k \\ &+ \langle (f_1 \hat{q}_2 + f_2 \hat{p}_2 - \langle f_1 \hat{q}_2 + f_2 \hat{p}_2 \rangle_k)^2 \rangle_k \end{aligned} \quad (2.12)$$

and similarly for  $\langle (\hat{X}(g, h) - \langle \hat{X}(g, h) \rangle_k)^2 \rangle_k$ . We thus have

$$\begin{aligned} & [\langle (\hat{X}(d, f) - \langle \hat{X}(d, f) \rangle_k)^2 \rangle_k + \langle (\hat{X}(g, h) - \langle \hat{X}(g, h) \rangle_k)^2 \rangle_k] \\ &= [\langle (\hat{X}(d, 0) - \langle \hat{X}(d, 0) \rangle_k)^2 \rangle_k + \langle (\hat{X}(g, 0) - \langle \hat{X}(g, 0) \rangle_k)^2 \rangle_k \\ &+ \langle (\hat{X}(0, f) - \langle \hat{X}(0, f) \rangle_k)^2 \rangle_k + \langle (\hat{X}(0, h) - \langle \hat{X}(0, h) \rangle_k)^2 \rangle_k] \\ &\geq |d^T Jg| + |f^T Jh| \end{aligned} \quad (2.13)$$

which holds for any  $d \sim h$ . Here we used (2.9) for  $f = h = 0$  or  $d = g = 0$ . The equality sign holds only for

$$\begin{aligned} & [(d_1 \hat{q}_1 + d_2 \hat{p}_1 - \langle d_1 \hat{q}_1 + d_2 \hat{p}_1 \rangle_k) + i(g_1 \hat{q}_1 + g_2 \hat{p}_1 - \langle g_1 \hat{q}_1 + g_2 \hat{p}_1 \rangle_k)] \phi_k(q_1) = 0, \\ & [(f_1 \hat{q}_2 + f_2 \hat{p}_2 - \langle f_1 \hat{q}_2 + f_2 \hat{p}_2 \rangle_k) + i(h_1 \hat{q}_2 + h_2 \hat{p}_2 - \langle h_1 \hat{q}_2 + h_2 \hat{p}_2 \rangle_k)] \varphi_k(q_2) = 0, \end{aligned} \quad (2.14)$$

for suitable  $d \sim h$  with  $d^T Jg > 0$  and  $f^T Jh > 0$ .

We finally conclude from (2.6) and (2.13) for any separable density matrix

$$\begin{aligned} & \langle (\Delta \hat{X}(d, f))^2 \rangle + \langle (\Delta \hat{X}(g, h))^2 \rangle \\ & \geq \sum_k P_k [\langle (\hat{X}(d, f) - \langle \hat{X}(d, f) \rangle_k)^2 \rangle_k + \langle (\hat{X}(g, h) - \langle \hat{X}(g, h) \rangle_k)^2 \rangle_k] \\ & + |d^T Jg| + |f^T Jh| \end{aligned} \quad (2.15)$$

for any  $d \sim h$ .

We next define the variables  $(\hat{\xi}_\alpha) = (\hat{q}_1, \hat{p}_1, \hat{q}_2, \hat{p}_2)$  and the  $4 \times 4$  correlation matrix  $V$  by

$$V = (V_{\alpha\beta}), \quad V_{\alpha\beta} = \frac{1}{2} \langle \Delta \hat{\xi}_\alpha \Delta \hat{\xi}_\beta + \Delta \hat{\xi}_\beta \Delta \hat{\xi}_\alpha \rangle = \frac{1}{2} \langle \{ \Delta \hat{\xi}_\alpha, \Delta \hat{\xi}_\beta \} \rangle \quad (2.16)$$

with  $\Delta \hat{\xi}_\alpha = \hat{\xi}_\alpha - \langle \hat{\xi}_\alpha \rangle$ , which can be written in the form

$$V = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix} \quad (2.17)$$

where  $A$  and  $B$  are  $2 \times 2$  real symmetric matrices and  $C$  is a  $2 \times 2$  real matrix. We also define

$$\tilde{V} = (\tilde{V}_{\alpha\beta}), \quad \tilde{V}_{\alpha\beta} = \sum_k P_k \langle \Delta \hat{\xi}_\alpha \rangle_k \langle \Delta \hat{\xi}_\beta \rangle_k \quad (2.18)$$

and

$$\tilde{V} = \begin{pmatrix} \tilde{A} & \tilde{C} \\ \tilde{C}^T & \tilde{B} \end{pmatrix} \quad (2.19)$$

where  $\tilde{A}$  and  $\tilde{B}$  are  $2 \times 2$  real symmetric matrices and  $\tilde{C}$  is a  $2 \times 2$  real matrix. Both of  $V$  and  $\tilde{V}$  are non-negative. This quantity  $\tilde{V}$  plays a central role in the P-representation.

The basic relation (2.15) for separable states is then written as

$$\begin{aligned} & d^T A d + f^T B f + 2d^T C f + g^T A g + h^T B h + 2g^T C h \\ & \geq d^T \tilde{A} d + f^T \tilde{B} f + 2d^T \tilde{C} f + g^T \tilde{A} g + h^T \tilde{B} h + 2g^T \tilde{C} h \\ & + |d^T J g| + |f^T J h|. \end{aligned} \quad (2.20)$$

while the Kennard relation for general states is written as

$$\begin{aligned} & d^T A d + f^T B f + 2d^T C f + g^T A g + h^T B h + 2g^T C h \\ & \geq d^T \tilde{A} d + f^T \tilde{B} f + 2d^T \tilde{C} f + g^T \tilde{A} g + h^T \tilde{B} h + 2g^T \tilde{C} h \\ & + |d^T J g + f^T J h|. \end{aligned} \quad (2.21)$$

Note the difference between  $|d^T J g| + |f^T J h|$  and  $|d^T J g + f^T J h|$ .

The antisymmetric commutator parts in

$$\langle \Delta \hat{\xi}_\alpha \Delta \hat{\xi}_\beta \rangle = \frac{1}{2} \langle \{ \Delta \hat{\xi}_\alpha, \Delta \hat{\xi}_\beta \} \rangle + \frac{1}{2} \langle [ \Delta \hat{\xi}_\alpha, \Delta \hat{\xi}_\beta ] \rangle \quad (2.22)$$

which may be added to  $A$  and  $B$  in (2.17) do not contribute to (2.20) since  $d \sim h$  are all real.

Under the  $S_1 \otimes S_2 \in Sp(2, R) \otimes Sp(2, R)$  transformations of  $(\hat{q}_1, \hat{p}_1)$  and  $(\hat{q}_2, \hat{p}_2)$ , respectively, we have

$$\begin{aligned} A &\rightarrow S_1 A S_1^T, & B &\rightarrow S_2 B S_2^T, & C &\rightarrow S_1 C S_2^T \\ \tilde{A} &\rightarrow S_1 \tilde{A} S_1^T, & \tilde{B} &\rightarrow S_2 \tilde{B} S_2^T, & \tilde{C} &\rightarrow S_1 \tilde{C} S_2^T \end{aligned} \quad (2.23)$$

which is equivalent to the transformation

$$d \rightarrow S_1^T d, \quad f \rightarrow S_2^T f, \quad g \rightarrow S_1^T g, \quad h \rightarrow S_2^T h \quad (2.24)$$

in (2.20) if one recalls  $J = S_1 J S_1^T$ ,  $J = S_2 J S_2^T$ ; the inequality (2.20), which is valid for any  $d \sim h$ , holds after the transformation (2.24) and in this sense (2.20) is invariant under the above  $Sp(2, R) \otimes Sp(2, R)$ . To be precise, we do not use any property of the wave function under  $Sp(2, R) \otimes Sp(2, R)$ , and thus our  $Sp(2, R) \otimes Sp(2, R)$  transformation is rather defined by (2.23) for given constant matrices  $A$ ,  $B$  and  $C$ .

The difference between the two expressions in (2.20) and (2.21) appears when one replaces  $B$  and  $C$  by  $S_3^T B S_3$  and  $C S_3$  (and also  $\tilde{B}$  and  $\tilde{C}$  by  $S_3^T \tilde{B} S_3$  and  $\tilde{C} S_3$ ), respectively, with

$$S_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2.25)$$

One can undo the replacements in the first expression (2.20) by transformations  $f \rightarrow S_3 f$  and  $h \rightarrow S_3 h$ , whereas it leads to  $|d^T J g - f^T J h|$  in the second Kennard relation (2.21). The separability condition thus demands that the Kennard relation should hold both for the original system and for the system with the replacements of  $B$  and  $C$  by  $S_3^T B S_3$  and  $C S_3$ , respectively, which may *a priori* be unphysical for inseparable systems. By using  $S_3$ , one can adjust the signature of  $\det C$  at one's will [5].

It is also useful to consider the separability condition (2.20) with subsidiary conditions  $g = J^T d$  and  $h = J^T f$ ,

$$\begin{aligned} &d^T A d + f^T B f + 2d^T C f + d^T J A J^T d + f^T J B J^T f + 2d^T J C J^T f \\ &\geq d^T \tilde{A} d + f^T \tilde{B} f + 2d^T \tilde{C} f + d^T J \tilde{A} J^T d + f^T J \tilde{B} J^T f + 2d^T J \tilde{C} J^T f \\ &+ (d^T d + f^T f) \end{aligned} \quad (2.26)$$

and with subsidiary conditions  $g = J^T d$  and  $h = -J^T f$

$$\begin{aligned} &d^T A d + f^T B f + 2d^T C f + d^T J A J^T d + f^T J B J^T f - 2d^T J C J^T f \\ &\geq d^T \tilde{A} d + f^T \tilde{B} f + 2d^T \tilde{C} f + d^T J \tilde{A} J^T d + f^T J \tilde{B} J^T f - 2d^T J \tilde{C} J^T f \\ &+ (d^T d + f^T f). \end{aligned} \quad (2.27)$$

For general inseparable states in (2.21), we have the first condition (2.26) only if one wants to keep  $(d^T d + f^T f)$  on the right-hand side in this form. The basic  $Sp(2, R) \otimes Sp(2, R)$  invariance of uncertainty relations (2.20) and (2.21) is lost in these conditions (2.26) and (2.27) with subsidiary conditions, but they have applications in the analysis of the P-representation.

### 3 Separability and P-representation in Gaussian states

#### 3.1 General analysis

One can bring any given  $V$  in (2.17) by  $Sp(2, R) \otimes Sp(2, R)$  transformations to the standard form [4, 5] (see also Appendix A)

$$V_0 = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix}. \quad (3.1)$$

One may understand the relations (2.20) and (2.21) ( and also (2.26) and (2.27)) in two different ways:

- (i) In the first interpretation, one may understand these relations (2.20) and (2.21) as an infinite set of uncertainty relations (and their variants) for any given constants  $d \sim h$ . In this interpretation, the relations (2.26) and (2.27) correspond to the ones used by DGCZ [4] if one chooses  $d$  and  $f$  suitably.
- (ii) In the second interpretation of the relations (2.20) and (2.21), one may understand these relations holding for any choice of  $d \sim h$  and thus imposing constraints on the allowed ranges of the elements  $a, b, c_1, c_2$  of the standard form of  $V_0$  in (3.1), for example. We adopt this second interpretation, which was also adopted by Simon [5]. In this interpretation, the full relation (2.20) is more restrictive than the relations (2.26) and (2.27) with the *subsidiary conditions* on  $d \sim h$ . In other words, the elements  $a, b, c_1, c_2$  which satisfy (2.20) automatically satisfy (2.26) and (2.27), but not the other way around. In our analysis below, we interpret these relations as constraints on  $c_1, c_2$  for fixed  $a, b$ .

The separability criterion is given by (2.20). On the other hand, the P-representation depends on the condition (see Appendix B)

$$V \geq \frac{1}{2}I \quad (3.2)$$

namely

$$d^T A d + f^T B f + 2d^T C f \geq \frac{1}{2}(d^T d + f^T f) \quad (3.3)$$

for any  $d \sim f$ . By using a special property of the P-representation, namely, a special property of the coherent state, one can also write (3.2) as

$$P^{-1} = \begin{pmatrix} \tilde{A} & \tilde{C} \\ \tilde{C}^T & \tilde{B} \end{pmatrix} = V - \frac{1}{2}I \geq 0 \quad (3.4)$$

or

$$\begin{aligned} & d^T \tilde{A} d + f^T \tilde{B} f + 2d^T \tilde{C} f \\ &= d^T A d + f^T B f + 2d^T C f - \frac{1}{2}(d^T d + f^T f) \geq 0 \end{aligned} \quad (3.5)$$

for any  $d \sim f$ . Here  $P^{-1}$  agrees with  $\tilde{V}$  in (2.19). See Appendix B.

We first note that the P-representation implies the separability condition, since (3.5) also implies

$$\begin{aligned} & g^T \tilde{A} g + h^T \tilde{B} h + 2g^T \tilde{C} h \\ &= g^T A g + h^T B h + 2g^T C h - \frac{1}{2}(g^T g + h^T h) \geq 0 \end{aligned} \quad (3.6)$$

and thus adding these two relations (3.5) and (3.6), we have

$$\begin{aligned} & d^T A d + f^T B f + 2d^T C f + g^T A g + h^T B h + 2g^T C h \\ &= d^T \tilde{A} d + f^T \tilde{B} f + 2d^T \tilde{C} f + g^T \tilde{A} g + h^T \tilde{B} h + 2g^T \tilde{C} h \\ &= \frac{1}{2}(d^T d + f^T f) + \frac{1}{2}(g^T g + h^T h). \end{aligned} \quad (3.7)$$

When one combines this relation with

$$\begin{aligned} \frac{1}{2}(d^T d + f^T f) + \frac{1}{2}(g^T g + h^T h) &\geq \sqrt{(d^T d + f^T f)(g^T g + h^T h)} \\ &= \sqrt{(d^T d + f^T f)(g^T J^T J g + h^T J^T J h)} \\ &\geq |d^T J g| + |f^T J h|, \end{aligned} \quad (3.8)$$

one reproduces the separability condition (2.20). This is natural since the P-representation is in fact separable.

But the inverse is not obvious. The separability condition (2,20) is invariant under  $Sp(2, R) \otimes Sp(2, R)$  in (2.23), whereas the P-representation condition (3.2) or (3.3) is not invariant under  $Sp(2, R) \otimes Sp(2, R)$  by noting that

$$d^T S_1^T S_1 d + f^T S_2^T S_2 f \neq d^T d + f^T f \quad (3.9)$$



in general. In this sense these two conditions cannot be equivalent to each other. One may write the condition for P-representation as

$$V - \frac{1}{2}SS^T \geq 0 \quad (3.10)$$

for a *suitable but arbitrary*  $S \in Sp(2, R) \otimes Sp(2, R)$ ; this relation implies the ordinary P-representation condition  $S^{-1}V(S^{-1})^T - \frac{1}{2}I \geq 0$  for a suitable  $Sp(2, R) \otimes Sp(2, R)$  transformed  $S^{-1}V(S^{-1})^T$ . Written in the form (3.10), the condition for P-representation has a formally invariant meaning in the following sense. For any  $S_1 \in Sp(2, R) \otimes Sp(2, R)$ , we have

$$S_1VS_1^T - \frac{1}{2}S_1SS^TS_1^T \geq 0 \quad (3.11)$$

which is written as

$$V' - \frac{1}{2}S'(S')^T \geq 0 \quad (3.12)$$

with  $V' = S_1VS_1^T$  and  $S' = S_1S \in Sp(2, R) \otimes Sp(2, R)$ .

### 3.2 A new explicit proof

We here present an explicit proof of the separability criterion for continuous variable two-party Gaussian systems. Our explicit construction gives the formulas of squeezing parameters, which establish the equivalence of the separability condition with the P-representation condition, in terms of the parameters of the standard form of the correlation matrix (3.1).

When one regards the separability condition as a constraint on the range of  $|c_1|$  and  $|c_2|$  in the standard form  $V_0$  (3.1), it is written as

$$\begin{aligned} 4(ab - c_1^2)(ab - c_2^2) &\geq (a^2 + b^2) + 2|c_1c_2| - \frac{1}{4}, \\ \sqrt{(2a - 1)(2b - 1)} &\geq |c_1| + |c_2| \end{aligned} \quad (3.13)$$

together with  $a \geq 1/2$  and  $b \geq 1/2$ . The conditions  $a \geq 1/2$  and  $b \geq 1/2$  are respectively derived by setting  $f = h = 0$  and  $d = g = 0$  in (2.20). The first relation in (3.13), which was derived by Simon [5], corresponds to

$$4\det[V_0 + \frac{i}{2} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}] \geq 0 \quad (3.14)$$

up to a transformation  $S_3$  in (2.25), and thus it is manifestly invariant under  $Sp(2, R) \otimes Sp(2, R)$ . The second condition in (3.13) is derived by the weaker conditions in (2.26) and (2.27) for the standard representation  $V_0$ , and it is used to exclude the nonsensical solutions of (3.13) with  $c_1^2 \rightarrow \infty$  and  $c_2^2 \rightarrow \infty$  for fixed  $a$  and  $b$ . The conditions (3.13) are equivalent to (2.20).

The separability condition (3.13) is explicitly solved as

$$\begin{aligned} c_1^2 &\leq \frac{1}{4t^2} \{[2ab(1+t^2) + t] - 2\sqrt{a^2b^2(1-t^2)^2 + t(a+bt)(at+b)}\}, \\ c_2^2 &\leq \frac{1}{4} \{[2ab(1+t^2) + t] - 2\sqrt{a^2b^2(1-t^2)^2 + t(a+bt)(at+b)}\} \end{aligned} \quad (3.15)$$

for

$$0 \leq t \equiv |c_2|/|c_1| \leq 1 \quad (3.16)$$

where we choose  $|c_2| \leq |c_1|$  without loss of generality.

On the other hand, one may choose  $S$  in (3.10) as

$$S(r_1, r_2)S^T(r_1, r_2) = \begin{pmatrix} 1/r_1 & 0 & 0 & 0 \\ 0 & r_1 & 0 & 0 \\ 0 & 0 & 1/r_2 & 0 \\ 0 & 0 & 0 & r_2 \end{pmatrix} \quad (3.17)$$

with suitably chosen  $r_1 \geq 1$  and  $r_2 \geq 1$ . By choosing the standard form of  $V_0$  in (3.1), the eigenvalues of  $V_0 - \frac{1}{2}S(r_1, r_2)S^T(r_1, r_2)$  are given by

$$\begin{aligned} (\lambda_1)_\pm &= \frac{1}{2} \left\{ \left( a - \frac{1}{2r_1} \right) + \left( b - \frac{1}{2r_2} \right) \pm \sqrt{\left( \left( a - \frac{1}{2r_1} \right) - \left( b - \frac{1}{2r_2} \right) \right)^2 + 4c_1^2} \right\}, \\ (\lambda_2)_\pm &= \frac{1}{2} \left\{ \left( a - \frac{1}{2}r_1 \right) + \left( b - \frac{1}{2}r_2 \right) \pm \sqrt{\left( \left( a - \frac{1}{2}r_1 \right) - \left( b - \frac{1}{2}r_2 \right) \right)^2 + 4c_2^2} \right\} \end{aligned} \quad (3.18)$$

The P-representation exists if  $(\lambda_1)_\pm \geq 0$  and  $(\lambda_2)_\pm \geq 0$ , namely, if the following two conditions are simultaneously satisfied

$$\begin{aligned} \left( a - \frac{1}{2r_1} \right) \left( b - \frac{1}{2r_2} \right) &\geq c_1^2, \\ \left( a - \frac{1}{2}r_1 \right) \left( b - \frac{1}{2}r_2 \right) &\geq c_2^2 \end{aligned} \quad (3.19)$$

together with  $a \geq \frac{1}{2}$ ,  $b \geq \frac{1}{2}$ ,  $(a - \frac{1}{2r_1}) + (b - \frac{1}{2r_2}) \geq 0$ , and  $(a - \frac{1}{2}r_1) + (b - \frac{1}{2}r_2) \geq 0$ .

When one regards (3.15) and (3.19) as constraints on the pair of variables

$$(c_1^2, c_2^2) \quad (3.20)$$

for given  $a$  and  $b$ , the P-representation condition is more restrictive than the separability conditions, namely, the set of points  $(c_1^2, c_2^2)$  allowed by the P-representation condition (3.19) always satisfy the separability condition (3.15). To be precise, we are working on the line defined by  $t^2 = c_2^2/c_1^2$ . We thus expect that these two conditions can coincide only for the extremal value of the P-representation condition (3.19) with respect to  $r_1$  and  $r_2$  with fixed  $t$ . We show that this is indeed the case.

We thus want to prove

$$\begin{aligned} & \left(a - \frac{1}{2r_1}\right)\left(b - \frac{1}{2r_2(t, r_1)}\right) \\ &= \frac{1}{t^2}\left[\left(a - \frac{1}{2}r_1\right)\left(b - \frac{1}{2}r_2(t, r_1)\right)\right] \\ &= \frac{1}{4t^2}\{[2ab(1+t^2) + t] - 2\sqrt{a^2b^2(1-t^2)^2 + t(a+bt)(at+b)}\} \end{aligned} \quad (3.21)$$

for a suitable  $1 \leq r_1 \leq 2a$  (and  $1 \leq r_2 \leq 2b$ ) for any given  $0 \leq t \leq 1$  by regarding  $r_2$  as a function of  $r_1$  and  $t$ . By this way we establish that the separability condition (3.15) agrees with the P-representation condition (3.19) with a suitable  $Sp(2, R) \otimes Sp(2, R)$  transformation.

We start with the equality in the left-hand side of (3.21)

$$\left(a - \frac{1}{2r_1}\right)\left(b - \frac{1}{2r_2(t, r_1)}\right) = \frac{1}{t^2}\left(a - \frac{1}{2}r_1\right)\left(b - \frac{1}{2}r_2(t, r_1)\right) \quad (3.22)$$

and take the derivative of the both hand sides with respect to  $r_1$  with fixed  $t$ . We then have

$$\begin{aligned} & \left[\frac{1}{2r_1^2}\left(b - \frac{1}{2r_2(t, r_1)}\right) + \left(a - \frac{1}{2r_1}\right)\frac{1}{2r_2^2(t, r_1)}\frac{\partial r_2}{\partial r_1}\right] \\ &= \frac{1}{t^2}\left[-\frac{1}{2}\left(b - \frac{1}{2}r_2(t, r_1)\right) - \left(a - \frac{1}{2}r_1\right)\frac{1}{2}\frac{\partial r_2}{\partial r_1}\right] \end{aligned} \quad (3.23)$$

which is solved as

$$\frac{\partial r_2}{\partial r_1} = -\frac{\left(b - \frac{1}{2}r_2(t, r_1)\right) + \frac{t^2}{r_1^2}\left(b - \frac{1}{2r_2(t, r_1)}\right)}{\left(a - \frac{1}{2}r_1\right) + \frac{t^2}{r_2^2(t, r_1)}\left(a - \frac{1}{2r_1}\right)}. \quad (3.24)$$

We next consider the stationary point (or extremal) of

$$\left(a - \frac{1}{2}r_1\right)\left(b - \frac{1}{2}r_2(t, r_1)\right) \quad (3.25)$$

with fixed  $t$ , namely

$$-\frac{1}{2}(b - \frac{1}{2}r_2(t, r_1)) - (a - \frac{1}{2}r_1)\frac{1}{2}\frac{\partial r_2}{\partial r_1} = 0. \quad (3.26)$$

This relation combined with (3.24) gives rise to

$$(b - \frac{1}{2}r_2(t, r_1))(a - \frac{1}{2}r_1)\frac{1}{r_2^2} = (a - \frac{1}{2}r_1)(b - \frac{1}{2r_2(t, r_1)})\frac{1}{r_1^2} \quad (3.27)$$

The relations (3.22) and (3.27) give

$$r_1 = \frac{\frac{r_2}{t}a + \frac{1}{2}}{a + \frac{1}{2}\frac{r_2}{t}}, \quad r_2 = \frac{\frac{r_1}{t}b + \frac{1}{2}}{b + \frac{1}{2}\frac{r_1}{t}} \quad (3.28)$$

which are symmetric in  $r_1$  and  $r_2$ . The relations in (3.28) are solved as

$$\begin{aligned} r_1 &= \frac{1}{at + b} \{ab(1 - t^2) + \sqrt{a^2b^2(1 - t^2)^2 + t(a + bt)(at + b)}\}, \\ r_2 &= \frac{1}{a + bt} \{ab(1 - t^2) + \sqrt{a^2b^2(1 - t^2)^2 + t(a + bt)(at + b)}\} \end{aligned} \quad (3.29)$$

with  $0 \leq t = |c_2|/|c_1| \leq 1$ , which determine the squeezing parameters.

We see from (3.29) that

$$r_1 = r_2 = 1 \quad (3.30)$$

for  $t = 1$ , and

$$r_1 = 2a, \quad r_2 = 2b \quad (3.31)$$

for  $t = 0$ . One can also confirm

$$\begin{aligned} \infty > \frac{r_1}{t} &= \frac{a + bt}{-ab(1 - t^2) + \sqrt{a^2b^2(1 - t^2)^2 + t(a + bt)(at + b)}} \geq 1, \\ \infty > \frac{r_2}{t} &= \frac{at + b}{-ab(1 - t^2) + \sqrt{a^2b^2(1 - t^2)^2 + t(a + bt)(at + b)}} \geq 1 \end{aligned} \quad (3.32)$$

by noting  $t(at + b) \leq (a + bt)$  and  $t(a + bt) \leq (at + b)$  for  $0 \leq t \leq 1$  and the triangle inequality. By recalling (3.28), we thus conclude

$$2a \geq r_1 \geq 1, \quad 2b \geq r_2 \geq 1 \quad (3.33)$$

for  $a \geq \frac{1}{2}$  and  $b \geq \frac{1}{2}$ .

We finally evaluate by using  $r_1$  and  $r_2$  in (3.29)

$$\begin{aligned}
& \frac{1}{t^2} \left( a - \frac{1}{2} r_1 \right) \left( b - \frac{1}{2} r_2 \right) \\
&= \frac{1}{4t^2} \left[ a + at \left( \frac{a+bt}{at+b} \right) - \frac{\sqrt{a^2 b^2 (1-t^2)^2 + t(a+bt)(at+b)}}{at+b} \right] \\
&\times \left[ b + bt \left( \frac{at+b}{a+bt} \right) - \frac{\sqrt{a^2 b^2 (1-t^2)^2 + t(a+bt)(at+b)}}{a+bt} \right] \\
&= \frac{1}{4t^2} \{ [2ab(1+t^2) + t] - 2\sqrt{a^2 b^2 (1-t^2)^2 + t(a+bt)(at+b)} \} \quad (3.34)
\end{aligned}$$

which is a remarkable identity. This relation establishes (3.21), namely, the fact that the boundaries of the conditions for separability and P-representation coincide for any  $0 \leq t = |c_2|/|c_1| \leq 1$ .

Our explicit construction proves that the P-representation condition (3.19) with suitably chosen  $S(r_1, r_2) \in Sp(2, R) \otimes Sp(2, R)$ , where  $1 \leq r_1 \leq 2a$  and  $1 \leq r_2 \leq 2b$ , is equivalent to the separability condition (3.15) for any  $0 \leq t = |c_2|/|c_1| \leq 1$ , and thus the separability condition (3.15) is a necessary and sufficient separability criterion for two-party Gaussian systems. Our formulas of  $r_1$  and  $r_2$  in (3.29) give the explicit expressions of squeezing parameters to achieve the above equivalence in terms of the parameters of the standard form of the correlation matrix (3.1).

## 4 Comparison with the past proofs

### 4.1 Proof of Duan, Giedke, Cirac and Zoller

The analysis of Duan, Giedke, Cirac and Zoller (DGCZ) [4] starts with the constraint

$$\frac{\left( \frac{2a}{r_1} - 1 \right)}{(2ar_1 - 1)} = \frac{\left( \frac{2b}{r_2} - 1 \right)}{(2br_2 - 1)} \quad (4.1)$$

which is written as

$$\frac{\left( \frac{n}{r_1} - 1 \right)}{(nr_1 - 1)} = \frac{\left( \frac{m}{r_2} - 1 \right)}{(mr_2 - 1)} \quad (4.2)$$

by noting  $2a = n$  and  $2b = m$  in their notation of the  $Sp(2, R) \otimes Sp(2, R)$  transformed correlation matrix

$$M = \begin{pmatrix} nr_1 & 0 & c\sqrt{r_1 r_2} & 0 \\ 0 & n/r_1 & 0 & c'/\sqrt{r_1 r_2} \\ c\sqrt{r_1 r_2} & 0 & mr_2 & 0 \\ 0 & c'/\sqrt{r_1 r_2} & 0 & m/r_2 \end{pmatrix} \quad (4.3)$$

with  $n > m \geq 1$  and  $|c| \geq |c'| > 0$ . Their normalization corresponds to  $M = 2V$ .

One can rewrite (4.2) as  $r_2(mr_2 - 1) = X(r_1)(m - r_2)$  and solve this quadratic equation in  $r_2$  in the form

$$r_2(r_1)_\pm = \frac{1 - X \pm \sqrt{(1 - X)^2 + 4m^2X}}{2m} \quad (4.4)$$

where we defined

$$X(r_1) = \frac{(nr_1 - 1)}{(\frac{n}{r_1} - 1)} = \frac{r_1(nr_1 - 1)}{n - r_1} \quad (4.5)$$

which assumes  $X(1) = 1$ ,  $X(n - \epsilon) = \infty$ ,  $X(n + \epsilon) = -\infty$  and  $X(\infty) = -\infty$ . Here  $\epsilon$  is an infinitesimal positive quantity which is eventually set to 0. One thus finds

$$\begin{aligned} r_2(1)_+ &= 1, & r_2(1)_- &= -1, \\ r_2(n - \epsilon)_+ &= m - \epsilon, & r_2(n - \epsilon)_- &= -\infty, \\ r_2(n + \epsilon)_+ &= \infty, & r_2(n + \epsilon)_- &= m + \epsilon, \\ r_2(\infty)_+ &= \infty, & r_2(\infty)_- &= m \end{aligned} \quad (4.6)$$

and thus the solution has a rather involved branch structure <sup>1</sup>.

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<sup>1</sup> By writing (4.5) as

$$X(r_1) \equiv \frac{r_1(nr_1 - 1)}{n - r_1} = -[n(r_1 - n) + \frac{n(n^2 - 1)}{r_1 - n}] - (2n^2 - 1)$$

one can show  $-\infty < X(r_1) \leq -[n + \sqrt{n^2 - 1}]^2$  for  $n < r_1 < \infty$  and the upper bound is achieved at  $r_1 = n + \sqrt{n^2 - 1}$ . Since  $X > 0$  for  $1 \leq r_1 \leq n$ , one can confirm that the content inside the square root in (4.4)

$$(1 - X)^2 + 4m^2X = \left(X + (m + \sqrt{m^2 - 1})^2\right) \left(X + (m - \sqrt{m^2 - 1})^2\right)$$

is positive definite for  $1 \leq r_1 < \infty$  when  $n > m \geq 1$ . Consequently the solutions  $r_2(r_1)_\pm$  in (4.4) are real, and  $r_2(r_1)_+$  for  $1 \leq r_1 \leq n$  and  $r_2(r_1)_-$  for  $n < r_1 < \infty$  defines a continuous real function  $r_2(r_1)$  for  $1 \leq r_1 < \infty$ , which is all that is necessary in the analysis in [4]. Incidentally, the content inside the square root above becomes negative and thus  $r_2(r_1)_\pm$  become complex for  $m > n$  for some interval in  $n < r_1 < \infty$ . In particular for  $n = m$ ,  $r_2 = r_1$  is a solution.

When one rewrites (4.4) as

$$r_2(r_1)_\pm \equiv \frac{2m}{1 - \frac{1}{X} \pm \sqrt{(1 - \frac{1}{X})^2 + 4m^2 \frac{1}{X}}}$$

for  $X > 0$  by taking  $X$  inside the square root and continues this expression for negative  $X$  also, then the single branch  $r_2(r_1)_+$  covers the entire domain  $1 \leq r_1 < \infty$  for  $n > m$ . This is because one picks up an extra  $-$  sign when one takes negative  $X$  inside the square root.

One may next consider [4]

$$f(r_1) = \sqrt{r_1 r_2} |c| - \frac{|c'|}{\sqrt{r_1 r_2}} - [\sqrt{(nr_1 - 1)(mr_2 - 1)} - \sqrt{(\frac{n}{r_1} - 1)(\frac{m}{r_2} - 1)}] \quad (4.7)$$

which satisfies (if one chooses the first branch  $r_2(r_1)_+$  in (4.4))

$$f(1) = |c| - |c'| \geq 0. \quad (4.8)$$

They then use the bound [4]

$$|c| \leq \sqrt{n(m - \frac{1}{m})} = \sqrt{n(m - 1)\frac{m+1}{m}}, \quad (4.9)$$

which is derived from (2.26) and (2.27) by setting  $g = 0$ . One may now examine  $f(r_1)$  in (4.7) in the domain  $n < r_1 < \infty$  [4] by choosing the branch  $r_2(r_1)_-$  in (4.4) for which one can show  $m < r_2(r_1)_-$ . One then establishes  $f(\infty) \leq 0$  by using (4.9) as in [4].

One thus concludes a solution for  $f(r_1) = 0$  in the interval  $1 \leq r_1 < \infty$ , as shown in [4], for the set of states which satisfy the condition (4.9). Incidentally,

$$f(n) = \sqrt{nm} |c| - \frac{|c'|}{\sqrt{nm}} - \sqrt{(n^2 - 1)(m^2 - 1)}, \quad (4.10)$$

and thus if one can establish

$$\sqrt{nm} |c| - \frac{|c'|}{\sqrt{nm}} \leq \sqrt{(n^2 - 1)(m^2 - 1)}, \quad (4.11)$$

one then has  $f(n) \leq 0$ . So far we briefly summarized the analysis in [4] together with a comment on (4.10) and (4.11) which are used later.

We now examine the main issue of the separability condition, eq.(16) in [4],

$$a_0^2 \frac{n_1 + n_2}{2} + \frac{m_1 + m_2}{2a_0^2} - |c_1| - |c_2| \geq a_0^2 + \frac{1}{a_0^2} \quad (4.12)$$

with  $a_0^2 = \sqrt{\frac{m_1 - 1}{n_1 - 1}} = \sqrt{\frac{m_2 - 1}{n_2 - 1}}$ , which gives rise to

$$\sqrt{(m_1 - 1)(n_1 - 1)} \pm \sqrt{(m_2 - 1)(n_2 - 1)} \geq |c_1| + |c_2| \quad (4.13)$$

for  $1 \leq r_1 \leq n$  and  $n < r_1$ , respectively, if one recalls that  $n_2 < 1$  and  $m_2 < 1$  for  $n < r_1$  when (4.3) is written in the form

$$M = \begin{pmatrix} n_1 & 0 & c_1 & 0 \\ 0 & n_2 & 0 & c_2 \\ c_1 & 0 & m_1 & 0 \\ 0 & c_2 & 0 & m_2 \end{pmatrix}. \quad (4.14)$$

Note the appearance of the crucial  $\pm$  sign in (4.13) because of  $(n_2 - 1)\sqrt{\frac{m_2 - 1}{n_2 - 1}} = -|n_2 - 1|\sqrt{\frac{m_2 - 1}{n_2 - 1}} = -\sqrt{(m_2 - 1)(n_2 - 1)}$  for  $n_2 < 1$ , for example. It appears that this minus sign was overlooked in [4].

If  $f(r_1) = 0$  has a solution in the interval  $1 \leq r_1 \leq n$ , one can derive the condition for the P-representation (eq.(17) in [4])

$$\sqrt{(m_1 - 1)(n_1 - 1)} \geq |c_1|, \quad \sqrt{(m_2 - 1)(n_2 - 1)} \geq |c_2| \quad (4.15)$$

by combining the first relation in (4.13) with  $f(r_1) = 0$ , and the proof of the P-representation in [4] naturally goes through. On the other hand, if  $f(r_1) = 0$  has a solution in the interval  $n < r_1 < \infty$ , one finds that the separability condition (4.13) with  $n < r_1$  is inconsistent with  $f(r_1) = 0$  for  $|c_2| \neq 0$  since one then has

$$\sqrt{(m_1 - 1)(n_1 - 1)} - \sqrt{(m_2 - 1)(n_2 - 1)} = |c_1| - |c_2| \geq |c_1| + |c_2|. \quad (4.16)$$

This puzzling result for  $n < r_1$  may indicate that some essential information is missing to analyze the P-representation. This is indeed the case as shown below.

The condition  $M - I \geq 0$  of the P-representation in fact requires

$$(n_1 - 1) + (m_1 - 1) \geq 0, \quad (n_2 - 1) + (m_2 - 1) \geq 0 \quad (4.17)$$

in addition to (4.16), since the eigenvalues of  $M - I$  are given by

$$\begin{aligned} (\lambda_1)_\pm &= \frac{1}{2}[(n_1 - 1) + (m_1 - 1) \pm \sqrt{((n_1 - 1) - (m_1 - 1))^2 + 4c_1^2}], \\ (\lambda_2)_\pm &= \frac{1}{2}[(n_2 - 1) + (m_2 - 1) \pm \sqrt{((n_2 - 1) - (m_2 - 1))^2 + 4c_2^2}]. \end{aligned} \quad (4.18)$$

See (3.18) in a different representation. It is obvious that if (4.17) are not satisfied, at least one of the eigenvalues in (4.18) becomes negative. More intuitively,  $M - I \geq 0$  cannot be maintained for sufficiently small  $c_1$  and  $c_2$  if (4.17) are not satisfied. One can confirm that (4.15) is derived from the requirement  $(\lambda_{1,2})_\pm \geq 0$  in (4.18), which is equivalent to  $M - I \geq 0$ , only under the conditions (4.17). These extra conditions



(4.17) are missing in [4], and the condition  $(n_2 - 1) + (m_2 - 1) \geq 0$  is violated in the case with  $n < r_1 < \infty$  for which  $n_2 < 1$  and  $m_2 < 1$ , and thus no P-representation exists for  $n < r_1 < \infty$ <sup>2</sup>.

It has been argued in [4] that separability or inseparability is independent of squeezing, but when one replaces separability by the P-representation one finds that the P-representation is very sensitive to squeezing. No information about (4.17) is contained either in the separability condition (4.12)(eq.(16) in [4] in terms of EPR-like operators) or in their analysis of  $f(r_1) = 0$ , and in this sense one may conclude that the proof the P-representation in the original scheme of [4] is incomplete.

One may now recall that the exact separability condition (3.14) is  $Sp(2, R) \otimes Sp(2, R)$  invariant while the P-representation condition is not invariant as is shown in (3.9). The separability condition (3.15) is thus independent of squeezing parameters while the P-representation condition (3.19) explicitly depends on squeezing parameters. The squeezing is an auxiliary device to show that the P-representation condition combined with suitable squeezing is equivalent to the  $Sp(2, R) \otimes Sp(2, R)$  invariant separability condition. A salient feature of the analysis in [4] is that they use the separability condition (4.12) which depends on squeezing parameters.

To prove the P-representation starting with the separability condition, one needs to satisfy two conditions (4.17) and (4.15). A way to achieve this purpose in the framework of [4] is to show (4.15) by using (4.12) for the squeezing parameter in the range

$$1 \leq r_1 \leq n. \quad (4.19)$$

We then automatically ensure (4.17) by means of (4.2) (and (4.6)) which is always assumed in our analysis. The non-trivial task is to show (4.15). For this purpose, we start with the first relation in (4.13), which is derived from the separability condition (4.12) if (4.2) is satisfied for  $1 \leq r_1 \leq n$ . It is then confirmed that the first relation in (4.13) with  $r_1 = n$  (and thus  $r_2 = m$  because of (4.2))

$$\sqrt{(m^2 - 1)(n^2 - 1)} \geq |c_1| + |c_2| = \sqrt{nm}|c| + \frac{|c'|}{\sqrt{nm}} \quad (4.20)$$

ensures  $f(n) \leq 0$  in (4.10). We thus conclude that there exists a solution for  $f(r_1) = 0$  in the interval  $1 \leq r_1 \leq n$  for the state which satisfies the separability

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<sup>2</sup>It appears that for most cases in practice, the continuous variable states automatically satisfy Lemma 2 (standard form II) in [4], namely, the matrix  $M$  in (4.3) with the constraint (4.2) for a suitable  $r_1$  in the interval  $1 \leq r_1 < \infty$  which satisfies  $f(r_1) = 0$  in (4.7). It should be emphasized that standard form II in [4] as it stands is based on the quite weak condition (4.9) and thus it is valid for inseparable states also. The standard form II holds even for  $n < r_1$  but the P-representation does not exist for  $n < r_1$  and  $m < r_2$ , namely, for  $n_2 < 1$  and  $m_2 < 1$ .

condition (4.12) for any  $r_1$  in  $1 \leq r_1 \leq n$ . The proof of the condition  $M - I \geq 0$  for the P-representation is then complete. Namely, (4.15) (eq.(17) in [4]) together with (4.17) is established by combining the first inequality in (4.13) with  $f(r_1) = 0$  for any  $|c| \geq |c'|$  in (4.3). An important new ingredient of the present scheme compared to the original scheme in [4] is that the order of the analyses of  $f(r_1) = 0$  and the separability condition (4.12) is reversed and the separability condition (4.12) with  $1 \leq r_1 \leq n$  now plays a central role in the analysis of  $f(r_1) = 0$ . One can confirm that (4.15) is equivalent to (3.19) when converted into our notation.

We here add further comments on the scheme of DGCZ in view of our explicit construction in Section 3.

Firstly, it is interesting that their condition (4.2), of which origin is not clearly stated in [4], agrees with our extremal condition (3.27).

Secondly, it is shown that the weaker forms of the separability condition, (2.26) and (2.27), when applied to the representation (4.14) give rise to the condition

$$\sqrt{[(n_1 + n_2) - 2][(m_1 + m_2) - 2]} \geq |c_1| + |c_2|. \quad (4.21)$$

To be explicit, (2.26) gives

$$\begin{pmatrix} n_1 + n_2 & 0 & c_1 + c_2 & 0 \\ 0 & n_1 + n_2 & 0 & c_1 + c_2 \\ c_1 + c_2 & 0 & m_1 + m_2 & 0 \\ 0 & c_1 + c_2 & 0 & m_1 + m_2 \end{pmatrix} - 2I \geq 0$$

by taking  $M = 2V$  into account, and (2.27) gives

$$\begin{pmatrix} n_1 + n_2 & 0 & c_1 - c_2 & 0 \\ 0 & n_1 + n_2 & 0 & -c_1 + c_2 \\ c_1 - c_2 & 0 & m_1 + m_2 & 0 \\ 0 & -c_1 + c_2 & 0 & m_1 + m_2 \end{pmatrix} - 2I \geq 0.$$

One can also show

$$\sqrt{[(n_1 + n_2) - 2][(m_1 + m_2) - 2]} \geq \sqrt{[n_1 - 1][m_1 - 1]} + \sqrt{[n_2 - 1][m_2 - 1]} \quad (4.22)$$

where the equality holds only when the condition (4.2) is satisfied. This relation (4.22) is established by considering

$$f(x) = \sqrt{[n_1 + x(n_2 - n_1) - 1][m_1 + x(m_2 - m_1) - 1]} \quad (4.23)$$

with

$$\begin{aligned}
f''(x) &= -\frac{1}{4}[(n_1 - 1)(m_2 - 1) - (n_2 - 1)(m_1 - 1)]^2 \\
&\times [m_1 + x(m_2 - m_1)) - 1]^{-3/2}[n_1 + x(n_2 - n_1)) - 1]^{-3/2} < 0
\end{aligned} \tag{4.24}$$

except for (4.2), namely,

$$\frac{(n_2 - 1)}{(n_1 - 1)} = \frac{(m_2 - 1)}{(m_1 - 1)}$$

for which  $f''(x) = 0$ . By using the property of the convex function  $2f(1/2) \geq f(1) + f(0)$  one can establish (4.22); the condition  $1 \leq r_1 \leq n$  is sufficient to keep  $f(x)$  real for  $0 \leq x \leq 1$ . Their separability condition (4.13), which is derived from (4.21) when the equality in (4.22) holds, thus corresponds to the weaker form of the separability condition.

This fact suggests that under the extremal condition (4.2), the weaker forms of the separability condition, (2.16) and (2.27), are sufficient to ensure the P-representation if supplemented by an additional constraint  $1 \leq r_1 \leq n$ .

## 4.2 Proof of Simon

The analysis of the case  $c_1 c_2 \geq 0$  by Simon [5] is quite elegant. Starting with the standard form of  $V_0$  in (3.1) and applying a set of  $Sp(2, R) \otimes Sp(2, R)$  transformations, he arrives at the form of  $V$

$$V = \begin{pmatrix} ay^2x^2 & 0 & c_1y^2 & 0 \\ 0 & a/(y^2x^2) & 0 & c_2/y^2 \\ c_1y^2 & 0 & by^2/x^2 & 0 \\ 0 & c_2/y^2 & 0 & bx^2/y^2 \end{pmatrix} \tag{4.25}$$

which is also written as

$$V = \begin{pmatrix} ar_1 & 0 & c_1\sqrt{r_1r_2} & 0 \\ 0 & a/r_1 & 0 & c_2/\sqrt{r_1r_2} \\ c_1\sqrt{r_1r_2} & 0 & br_2 & 0 \\ 0 & c_2/\sqrt{r_1r_2} & 0 & b/r_2 \end{pmatrix} \tag{4.26}$$

by defining

$$r_1 = (xy)^2, \quad r_2 = y^2/x^2. \tag{4.27}$$

He uses the crucial condition

$$\frac{c_1}{ax^2 - b/x^2} = \frac{c_2}{a/x^2 - bx^2} \quad (4.28)$$

which allows the diagonalization of  $V$  by a  $Sp(4, R)$  transformation. This  $Sp(4, R)$  preserves the Kennard relation, and thus one can use the Kennard relation to show the P-representation.

The condition (4.28) is written as

$$x^4 = \frac{r_1}{r_2} = \frac{c_1 a + c_2 b}{c_2 a + c_1 b} = \frac{a + (c_2/c_1)b}{(c_2/c_1)a + b} \quad (4.29)$$

which implies (for  $a \geq b$ )

$$1 \leq \frac{r_1}{r_2} \leq \frac{a}{b}. \quad (4.30)$$

It is interesting that the condition (4.29) agrees with our explicit construction (3.29).

Simon [5] shows that  $V$  in (4.25) can be diagonalized by an  $Sp(4, R)$  transformation as  $V' = \text{diag}(\kappa_+, \kappa'_+, \kappa_-, \kappa'_-)$  with

$$\begin{aligned} \kappa_{\pm} &= \frac{1}{2}y^2\{ax^2 + b/x^2 \pm \sqrt{(ax^2 - b/x^2)^2 + 4c_1^2}\}, \\ \kappa'_{\pm} &= \frac{1}{2}y^{-2}\{a/x^2 + bx^2 \pm \sqrt{(a/x^2 - bx^2)^2 + 4c_2^2}\}. \end{aligned} \quad (4.31)$$

The equality of two smaller eigenvalues  $\kappa_- = \kappa'_-$  is ensured if

$$\begin{aligned} y^4 = r_1 r_2 &= \frac{a/x^2 + bx^2 - \sqrt{(a/x^2 - bx^2)^2 + 4c_2^2}}{ax^2 + b/x^2 - \sqrt{(ax^2 - b/x^2)^2 + 4c_1^2}} \\ &= \frac{a + b(r_1/r_2) - \sqrt{(a - b(r_1/r_2))^2 + 4c_2^2(r_1/r_2)}}{a(r_1/r_2) + b - \sqrt{(a(r_1/r_2) - b)^2 + 4c_1^2(r_1/r_2)}} \end{aligned} \quad (4.32)$$

and the Kennard's relation (uncertainty relation),  $\kappa_- \kappa'_- \geq 1/4$ , implies  $\kappa_- = \kappa'_- \geq 1/2$  which in turn gives rise to

$$\begin{aligned} (ar_1 - \frac{1}{2})(br_2 - \frac{1}{2}) &\geq c_1^2 r_1 r_2, \\ (a/r_1 - \frac{1}{2})(b/r_2 - \frac{1}{2}) &\geq c_2^2/(r_1 r_2) \end{aligned} \quad (4.33)$$

or equivalently

$$\begin{aligned} (a - \frac{1}{2r_1})(b - \frac{1}{2r_2}) &\geq c_1^2, \\ (a - \frac{1}{2}r_1)(b - \frac{1}{2}r_2) &\geq c_2^2. \end{aligned} \tag{4.34}$$

This last relation naturally agrees with the condition of the P-representation (3.19). The separability condition, which agrees with the Kennard's relation for  $c_1 c_2 > 0$ , thus ensures the P-representation. The case  $c_1 c_2 < 0$  is equally treated by replacing  $c_1$  and  $c_2$  by  $|c_1|$  and  $|c_2|$ , respectively [5].

Although we have not succeeded in proving (4.32) by using our explicit solution (3.29) due to technical complications, one can confirm that (4.29) combined with one of the relations (3.28) gives rise to our explicit solution (3.29). We thus believe that the solution given by Simon agrees with our explicit construction.

## 5 Conclusion

We have presented an elementary and explicit analysis of the separability criterion of continuous variable two-party Gaussian systems. In particular, we derived the explicit formulas of squeezing parameters, which establish the equivalence of the separability condition with the P-representation condition, in terms of the parameters of the standard form of the correlation matrix (or second moments). In the course of our analysis, we corrected the shortcomings of the past proof of DGCZ [4]. Our explicit construction also clarified the basic equivalence of the past seemingly quite different proofs of the separability criterion [4, 5] in the sense that both of the past proofs are closely related to the present explicitly constructed solution.

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## A Standard form of $V$

We recall the elements of  $Sp(2, R)$

$$S = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad S = \begin{pmatrix} x & 0 \\ 0 & \frac{1}{x} \end{pmatrix} \tag{A.1}$$

which satisfy  $SJS^T = J$ . One can bring  $V$  in (2.17) to the standard form

$$V = \begin{pmatrix} a & 0 & c_1 & 0 \\ 0 & a & 0 & c_2 \\ c_1 & 0 & b & 0 \\ 0 & c_2 & 0 & b \end{pmatrix} \quad (\text{A.2})$$

by suitable  $Sp(2, R) \otimes Sp(2, R)$  transformations [4, 5]; real symmetric  $A$  and  $B$  can be made diagonal by two-dimensional rotations with suitable parameters  $\theta$  in (A.1) and then applying the second elements in (A.1) with suitable parameters  $x$ ,  $A$  and  $B$  are made proportional to the unit matrix. After these transformations  $C$  remains real. By applying a suitable two-dimensional orthogonal transformation  $S_1 \otimes S_2$ , which is an element of  $Sp(2, R) \otimes Sp(2, R)$ , we can diagonalize  $C$

$$S_1 C S_2^T = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}. \quad (\text{A.3})$$

By this way we arrive at (A.2).

## B P-representation

We define the generating function of all the correlations (or moments) of dynamical variables by

$$\chi(\lambda, \eta) = \text{Tr}(\hat{\rho} \exp\{i(\lambda_1 \hat{q}_1 + \lambda_2 \hat{p}_1 + \eta_1 \hat{q}_2 + \eta_2 \hat{p}_2)\}) \quad (\text{B.1})$$

where  $\lambda_1 \sim \eta_2$  are real numbers. By expanding  $\chi(\lambda, \eta)$  in powers of  $\lambda_1 \sim \eta_2$ , one can generate all the moments of dynamical variables. Following the convention in this field, we define the Gaussian states by

$$\chi(\lambda, \eta) = \exp\left\{-\frac{1}{2}(\lambda_1, \lambda_2, \eta_1, \eta_2)V(\lambda_1, \lambda_2, \eta_1, \eta_2)^T\right\} \quad (\text{B.2})$$

where  $V$  is the correlation matrix in (2.17), namely, all the correlation functions are determined by the second moments. One can write (B.1) as

$$\chi(\lambda, \eta) = \text{Tr}(\hat{\rho} \exp\{i(\lambda^* \hat{a} + \lambda \hat{a}^\dagger + \eta^* \hat{b} + \eta \hat{b}^\dagger)\}) \quad (\text{B.3})$$

with

$$\begin{aligned} \hat{a} &= \frac{1}{\sqrt{2}}(\hat{q}_1 + i\hat{p}_1), & \hat{b} &= \frac{1}{\sqrt{2}}(\hat{q}_2 + i\hat{p}_2), \\ \lambda &= \frac{1}{\sqrt{2}}(\lambda_1 + i\lambda_2), & \eta &= \frac{1}{\sqrt{2}}(\eta_1 + i\eta_2) \end{aligned} \quad (\text{B.4})$$

The Gaussian state is called P-representable if the density matrix is written as

$$\hat{\rho} = \int d^2\alpha \int d^2\beta P(\alpha, \beta) |\alpha, \beta\rangle \langle \alpha, \beta| \quad (\text{B.5})$$

where  $|\alpha, \beta\rangle$  is the coherent state defined by

$$\hat{a}|\alpha, \beta\rangle = \alpha|\alpha, \beta\rangle, \quad \hat{b}|\alpha, \beta\rangle = \beta|\alpha, \beta\rangle, \quad \langle \alpha, \beta|\alpha, \beta\rangle = 1 \quad (\text{B.6})$$

or to be explicit

$$|\alpha, \beta\rangle = e^{\alpha\hat{a}^\dagger - \frac{1}{2}|\alpha|^2}|0\rangle \otimes e^{\beta\hat{b}^\dagger - \frac{1}{2}|\beta|^2}|0\rangle. \quad (\text{B.7})$$

Thus the P-representable states are separable.

By using the density matrix (B.5) in (B.3) and after normal ordering the exponential factor in (B.3), we have

$$\begin{aligned} \chi(\lambda, \eta) &= \int d^2\alpha \int d^2\beta P(\alpha, \beta) \exp\{i(\lambda^*\alpha + \lambda\alpha^* + \eta^*\beta + \eta\beta^*)\} \\ &\quad \times \exp\{-\frac{1}{2}(|\lambda|^2 + |\eta|^2)\} \end{aligned} \quad (\text{B.8})$$

or, if one combines this expression with (B.2) we have

$$\begin{aligned} &\exp\{-\frac{1}{2}(\lambda_1, \lambda_2, \eta_1, \eta_2)(V - \frac{1}{2}I)(\lambda_1, \lambda_2, \eta_1, \eta_2)^T\} \\ &= \int d^2\alpha \int d^2\beta P(\alpha, \beta) \exp\{i(\lambda_1\alpha_1 + \lambda_2\alpha_2 + \eta_1\beta_1 + \eta_2\beta_2)\} \end{aligned} \quad (\text{B.9})$$

with  $\alpha = (\alpha_1 + i\alpha_2)/\sqrt{2}$  and  $\beta = (\beta_1 + i\beta_2)/\sqrt{2}$ . Thus  $P(\alpha, \beta)$  in (B.9) is given by

$$P(\alpha, \beta) = \frac{\sqrt{\det P}}{4\pi^2} \exp\{-\frac{1}{2}(\alpha_1, \alpha_2, \beta_1, \beta_2)P(\alpha_1, \alpha_2, \beta_1, \beta_2)^T\} \quad (\text{B.10})$$

where

$$P^{-1} = V - \frac{1}{2}I \geq 0 \quad (\text{B.11})$$

which defines the condition for the P-representation.

The formula (B.9) indicates that the right-hand side generates the correlations of the form

$$\int d^2\alpha \int d^2\beta P(\alpha, \beta) \langle \alpha, \beta|\hat{a}^\dagger|\alpha, \beta\rangle \langle \alpha, \beta|\hat{a}|\alpha, \beta\rangle, \quad (\text{B.12})$$

for example, which may be compared to (2.18). By recalling (B.5), this relation shows that all the second moments in the right-hand side of (B.9) are given by  $\tilde{V}$  in (2.18) (if one chooses  $\langle \hat{a}^\dagger \rangle = \langle \hat{a} \rangle = \langle \hat{b}^\dagger \rangle = \langle \hat{b} \rangle = 0$ ). This property establishes the special relation (3.4) of the P-representation.

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